International Journal of Applied Mathematics
Statistical Sciences (IJAMSS)
ISSN(P): 2319-3972; ISSN(E): 2319-3980
Vol. 7, Issue 4, Jun - Jul 2018; 35-48

International Academy of Science, Engineering and Technology
IASET
Connecting Researchers; Nurturing Innovations

THE NUMERICAL SOLUTIONS OF MULTIPLE INTEGRALS BASED ON HAAR WAVELET, BLOCK-PULSE FUNCTION, CHEBYSHEV WAVELET AND SINC FUNCTION

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ABSTRACT

In this paper, present computational methods for solving numerical multiple integrals with constants and variables limits of integrations which are based on Haar wavelet, Block-pulse function, Chebyshev wavelet and Sinc-function methods. This approach is the generalization and improvement of these methods and compared with Gauss Legendre quadrature methods. The main advantages of the generalized method are its more efficient and simple applicability than the previous methods. An absolute and relative errors are estimated of multiple integrations are considered up to three dimensions. Finally, we also give some numerical examples to compare with existing methods and the benefits of proposed methods have to find their computation efficiency.

KEYWORDS: *Multiple Integral, Gauss Legendre Quadrature Method, Haar Wavelets, Block-Pulse Functions, Chebyshev Wavelet, Sinc Functions*

Article History

Received: 18 May 2018 | Revised: 22 May 2018 | Accepted: 30 May 2018

1. INTRODUCTION

Integrals and derivatives are fundamental to calculus. The integrals involve in mathematics, physics, and engineering applications. To find the solution of integrals is preferred clear procedures. The analytical solutions are not helpful for the real-time applications. Some integrals cannot be found with exact solutions analytically, if some special functions involving the integrals which is a challenge to compute, and difficult to finding the accurate solution is too time-consuming. This is the main reason to go for numerical methods for approximating integrals, nowadays to compute the integrations using digital electronic computers. So numerical integration is the best ways to find most approximate solutions of the integrations. In last decades, wavelets are developing tool in this area. There are different types of wavelets and approximating functions have been used in numerical approximations, some examples are the Haar wavelet [Maleknejad and Mirzaee (2005)][Aziz and Haq (2010)], Block-pulse function [Rabbani and Nouri (2010)], Chebyshev function [Yuanlu (2010)] [Rostami et al. (2012)] and Sinc function [Stenger (2012)]. The solution of numerical integral has been done up to triple integral using Haar wavelet [Ahmedov and bin Abd Sathar (2013)] [Aziz and Haq (2010)]. The single integral has been done using Block-pulse function, Chebyshev function and Sinc function [Rostami et al. (2012)].

In this paper, we generalized as N dimension integral $(N = 1,2,3, \ldots.)$ using the proposed methods, some examples of numerical multiple integrations (consider only single, double and triple integrals) with variable and constant limits are provided to show the precision of proposed methods and comparison between them. Those methods are compared with Gauss Legendre quadrature method. Absolute and relative errors are estimated of numerical multiple integrations. The advantage of proposed the methods has to find their computation efficiency.

In associate with this paper as follows. In section 2 as Numerical technique for single integrals of proposed methods, and the subsections 2.1 to 2.4 as method of numerical integrations based on Haar wavelet [Ahmedov and bin Abd Sathar (2013)] [Aziz and Haq (2010)], Block-pulse function [Rostami et al. (2012)], Chebyshev wavelet, Sinc function [Rostami et al. (2012)] respectively. In section 3, the method for generalized N dimension integrals. In section 4, has given numerical examples with results and section 5, is given results and discussion. Concluding remarks are given in the last section 6.

2. NUMERICAL INTEGRATION FOR SINGLE INTEGRALS

2.1. Haar Wavelet

The Haar wavelet is a sequence of rescaled square-shaped functions which together form a wavelet family or basis, and its scaling function $\phi_1(x)$ can be described as on the interval [a,b)

$$\phi_1(x) = \begin{cases} 1, \text{for } x \in [a,b) \\ 0, otherwise \end{cases} \tag{1}$$

The Haar wavelets mother wavelet function $\phi_2(x)$ can be described as on the interval [a, b) is given by

$$\phi_2(x) = \begin{cases} 1 & for\ x \in \left[a, \frac{a+b}{2}\right) \\ -1 & \text{for} x \in [\frac{a+b}{2}, b) \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

In general, Haar wavelet family defined for $x \in [a,b)$ except the scaling function can be described as

$$\phi_{\mathrm{i}}(x) = \begin{cases} 1 & for\ x \in [\alpha, \beta) \\ -1 & \text{for } x \in [\beta, \gamma) \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

for all $i = 3,4,5, \ldots\ldots, 2M$ where $\alpha = a + \frac{b-a}{m} k, \beta = a + \frac{b-a}{m}(k + 0.5), \gamma = a + \frac{b-a}{m}(k + 1)$ and $M = 2^J$, J be the maximum level of Haar resolutions and $k = 1,2, \ldots\ldots, 2M$

2.1.1. Method of Numerical Integrations Based on Haar Wavelet

Any function $f(x)$ which is square integrable function on the interval [a,b) can be expressed as an infinite sum of Haar wavelets

$$f(x) = \sum_{i=1}^{\infty} a_i \phi_i(x) \tag{4}$$

In practice, only the first 2M terms of the sum is considered as,

$$f(x) = \sum_{i=1}^{2M} a_i \phi_i(x) \tag{5}$$

Lemma 2.1 *The approximate value of the integral is*

$$\int_a^b f(x)dx = a_1(b-a) \tag{6}$$

Proof.

$$\int_a^b \phi_i(x)dx = 0$$

for all $i = 2,3,\ldots..$

and

$$\int_a^b \phi_1(x)dx = b-a$$

$$\int_a^b f(x)dx = \sum_{i=1}^{2M} a_i \int_a^b \phi_i(x)dx$$

$$= a_1(b-a)$$

In connection with the above equation Haar approximate only one co-efficient in the evaluation of definite integral to calculate the Haar co-efficient a_1, we consider as a nodal points as,

$$x_k = a + \frac{b-a}{2M}(k+0.5) \tag{7}$$

the discretized form of eqn. (5) can be written as

$$f(x_k) = \sum_{i=1}^{2M} a_i\phi_i(x_k) \tag{8}$$

Lemma 2.2 *The solution of the system for* a_k *is*

$$a_1 = \frac{1}{2M}\sum_{i=1}^{2M} f(x_k) \tag{9}$$

Proof. We can prove by an induction method on J, where $M = 2^J$, for $J = 0, M = 1$

$$f(x_1) = a_1 + a_2$$

$$f(x_2) = a_1 - a_2$$

therefore

$$a_1 = \frac{1}{2}[f(x_1) + f(x_2)]$$

this is true for $J = 0$, we assume that the lemma is true for upto $J = n-1$, let us consider $J = n$ the new system has contain 2^n equation involving 2^n variable, and replacing $2a_k$ by $a_k^{'}$ and $f(x_{2k-1}) + f(x_{2k})$ by $g(x_{2k})$, thus we have

$$a_k^{'} = \frac{1}{2.2^{n-1}}\sum_{k=1}^{2.2^{n-1}} g(x_k) \tag{10}$$

substituting $a_k^{'}$ and $g(x_{2k})$ values

$$a_k = \frac{1}{2.2^n}\sum_{k=1}^{2.2^n} f(x_k)$$

$$a_k = \frac{1}{2M}\sum_{k=1}^{2M} f(x_k)$$

the lemma is true for $J = n$, in particular

$$a_1 = \frac{1}{2M}\sum_{k=1}^{2M} f(x_k) \tag{11}$$

substituting equations (11) and (7) in eqn.(6), the formula for Haar wavelet is

$$\int_a^b f(x)dx = \frac{b-a}{2M}\sum_{k=1}^{2M} f\left(a + \frac{b-a}{2M}(k+0.5)\right) \tag{12}$$

2.2. Block-Pulse Function

Definition 2.1 *An m-set of BPF is defined as follows*

$$\phi_1(x) = \begin{cases} 1, \text{for } t \in \left[\frac{(i-1)T}{m}, \frac{iT}{m}\right) \\ 0, \text{otherwise} \end{cases} \tag{13}$$

with $t \in [0, T)$, $i = 1,2,3,\ldots\ldots, m$ and $h = \frac{T}{m}$

2.2.1. Method of Numerical Integrations Based on BPF

We consider the integral $\int_a^b f(x)dx$ by using $x = (b-a)t + a$, we have

$$\int_a^b f(x)dx = (b-a)\int_0^1 f((b-a)t + a)dt \tag{14}$$

Theorem 2.3 *The approximate value of the integral is*

$$\int_0^1 f(t)dt \approx \frac{1}{m}\sum_{i=1}^{m} f_i \tag{15}$$

Proof. the orthogonal property of BPF is the basis of expanding functions into their BPF series as

$$f(t) = \sum_{i=1}^{m} f_i \phi_i(t) \tag{16}$$

$$\int_0^1 f(t)dt = \sum_{i=1}^{m} f_i \int_0^1 \phi_i(t)dt$$

$$\approx \frac{1}{m}\sum_{i=1}^{m} f_i$$

Let us consider the nodal points as

$$t_k = \frac{2k-1}{2m}, k = 1,2,\ldots\ldots\ldots\ldots, m$$

$$f(t_k) = \sum_{i=1}^{m} f_i \phi_i(t_k) = f_k$$

The approximate value

$$\int_0^1 f(t)dt \approx \frac{1}{m}\sum_{i=1}^{m} f\left(\frac{2i-1}{2m}\right)$$

In general

$$\int_a^b f(t)dt \approx \frac{(b-a)}{m}\sum_{i=1}^{m} f\left(a + \frac{b-a}{m}(i-0.5)\right) \tag{17}$$

if $m = 2M, M = 2^J$, the BPF becomes Haar wavelet (see Tables)

2.3. Chebyshev Wavelet

Definition 2.2 *The Chebyshev wavelets,* $\phi_{n,m}(x), n = 1,2,\ldots\ldots, 2^{k-1}$ *and* $m = 0,1,\ldots\ldots, M-1$ *is defined on the* *interval [0,1) as,*

$$\phi_{n,m}(x) = \begin{cases} 2^{\frac{k}{2}}\tilde{T}_m(2^k x - 2n + 1), \text{for } x \in [\frac{(n-1)}{2^{k-1}}, \frac{n}{2^{k-1}}) \\ 0, otherwise \end{cases} \tag{18}$$

where

$$\tilde{T}_m(t) = \begin{cases} \frac{1}{\sqrt{\pi}}, \text{for } m = 0 \\ \sqrt{\frac{2}{\pi}} T_m(t), for\ m > 0 \end{cases}$$

and k is any any positive integer and m is the degree of Chebyshev polynomials of the first kind. Here, $T_m(t)$ is the Chebyshev polynomials of the first kind of degree m which is orthogonal with respect to the weight function $W(t) = \frac{1}{\sqrt{1-t^2}}$ on the interval [-1,1] and satisfy the following recursive formula

$$\begin{cases} T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t), m = 1,2,3, \ldots\ldots \\ T_0 = 1 and T_1(t) = t. \end{cases}$$

2.3.1. Method of Numerical Integrations Based on Chebyshev Wavelet

An integral over [a, b] must be changed into an integral over [0, 1]. This change of interval can be done in the following way

$$\int_a^b f(x)dx = (b-a)\int_0^1 f((b-a)t + a)dt \tag{19}$$

Theorem 2.4 *The approximate value of the integral is*

$$\int_0^1 f(x)dx \approx \frac{2^{1-\frac{k}{2}}}{\sqrt{\pi}} \sum_{n=1}^{2^{k-1}} [C_{n,0} + \sum_{l=1}^{M} \frac{\sqrt{2}}{1-4l^2} C_{n,2l}] \tag{20}$$

Proof. Any function $f(x)$, which is square integrable in the interval $x \in [0,1)$, can be expressed as

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} C_{n,m}\phi_{n,m}(x) x \in [a,b)$$

so we can approximate $f(x)$ as

$$f(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m}\phi_{n,m}(x) x \in [a,b)$$

$$\int_0^1 f(x)dx = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m} \int_0^1 \phi_{n,m}(x)dx \tag{21}$$

let us consider

$$\int_0^1 \phi_{n,m}(x)dx = 2^{\frac{k}{2}} \int_{\frac{(n-1)}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \tilde{T}_m(2^k x - 2n + 1)dx$$

$$2^{-\frac{k}{2}} \int_{-1}^{1} \tilde{T}_m(t)dt$$

since

$$\int_{-1}^{1} T_m(t)dt = \begin{cases} 0, m\ is\ odd \\ \frac{-2}{m^2-1}, m\ is\ even \end{cases}$$

we have

$$\int_0^1 \phi_{n,m}(x)dx = \begin{cases} \frac{2^{1-\frac{k}{2}}}{\sqrt{\pi}}1 & m = 0 \\ 0 & m \ is \ odd \\ \frac{2^{\frac{3-k}{2}}}{\sqrt{\pi}(1-m^2)} & m \ is \ even \end{cases}$$

substituting the above equations in eqn. (21), then we get eqn. (20).

Now to calculate the coefficient $C_{(n,0)}$ and $C_{(n,2l)}$ of Chebyshev wavelets, and consider the nodal points as

$$x_q = \frac{2q-1}{2^k M} \ q = 1,2,\ldots\ldots,2^{k-1}M$$

and

$$f(x_q) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m}\phi_{n,m}(x_q)$$

From the definition of Chebyshev wavelet

$$\frac{(n-1)}{2^{k-1}} \leq \frac{2q-1}{2^k M} < \frac{n}{2^{k-1}}$$

so

$$q = (n-1)M + i \ i = 1,2,3,\ldots\ldots,M$$

$$f(x_q) = f(\frac{2(n-1)M+2i-1}{2^k M})$$

and

$$\phi_{n,0}(x_q) = \frac{2^{\frac{k}{2}}}{\sqrt{\pi}}$$

$$\phi_{n,m}(x_q) = \frac{2^{\frac{1+k}{2}}}{\sqrt{\pi}}T_m(\frac{2i-1}{M} - 1)$$

The above system of equations changed to the below system of equations is given as

$$f(\frac{2(n-1)M+2i-1}{2^k M}) = \frac{2^{\frac{k}{2}}}{\sqrt{\pi}}C_{n,0} + \sum_{m=1}^{M-1} \frac{2^{\frac{1+k}{2}}}{\sqrt{\pi}}T_m(\frac{2i-1}{M} - 1)C_{n,m} \tag{22}$$

for all $i = 1,2,3,\ldots\ldots,M$

The coefficient $C_{(n,0)}$ and $C_{(n,2l)}$ can be easily calculated from the above system of equation.

2.4. Sinc Function

Definition 2.3 *The Sinc function is defined on the whole real line by*

$$sinc(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, \text{for } x \neq 0 \\ 1, \text{for } x = 0 \end{cases}$$

For any $h > 0$, the translated Sinc function with evenly spaced nodes are

$$S(J,h)(x) = sinc\left(\frac{x-Jh}{h}\right) \ J = 0, \pm 1, \pm 2, \ldots\ldots..$$

This is called J^{th} Sinc functions. If f is defined on the real line, then $h > 0$. The series

$$f(x) = \sum_{J=-\infty}^{\infty} f(Jh)S(J,h) \tag{23}$$

whenever the series f is converging. The expansion of f is called Whittaker cardinal expansion.

2.4.1. Method of Numerical Integrations Based on Sinc Function

In this case we take the limits (a,b), where $-\infty < a < b < \infty$

$$\phi(x) = ln(\frac{x-a}{b-x}) \tag{24}$$

$$\phi'(x) = \frac{b-a}{(x-a)(b-x)} \tag{25}$$

the basis function on (a,b) is given by

$$S(J,h)\phi(x) = sinc(\frac{\phi(x)-Jh}{h})$$

we consider the nodal point as $x_J \in (a,b)$

$$x_J = \phi^{-1}(Jh) = \frac{a+be^{Jh}}{1+e^{Jh}} \tag{26}$$

Theorem 2.5 *Let $L_\alpha(D)$ be the set of all analytic function, D be a simply connected domain in the complex plane, $\frac{f}{\phi'} \in L_\alpha(D)$ with $0 < \alpha \leq 1$ and $0 < d \leq \pi$, let N be the positive integer, and $h = (\frac{\pi d}{\alpha N})^{\frac{1}{2}}$ and there exist a positive constant C_2 which is independent of N, such that*

$$\left|\int_a^b f(t)dt - h\sum_{J=-N}^{N} \frac{f(x_J)}{\phi'(x_J)}\right| \leq C_2 e^{(-\pi d\alpha N)^{\frac{1}{2}}}$$

so the approximate integral based on Sinc function is

$$\int_a^b f(t)dt \approx h\sum_{J=-N}^{N} \frac{f(x_J)}{\phi'(x_J)} \tag{27}$$

3. METHOD FOR GENERALIZED N DIMENSION INTEGRALS

Consider the multiple integral with variable and constant limits of the type

$$\int_a^b \prod_{i=2}^{n} \int_{c(s_i)}^{d(s_i)} f(s_i, s_1)ds_i ds_1 \tag{28}$$

with $s_i = s_2, s_3, \ldots\ldots, s_n$

Applying the above proposed methods of weights (w_j) and the corresponding nodes (x_j) in the above equation to the inner integrals and by treating the outer integral variable s_1 as constant, the above equation can be written as

$$\int_a^b \prod_{i=2}^{n} \int_{c(s_i)}^{d(s_i)} f(s_i, s_1)ds_i ds_1 \approx \int_a^b \prod_{i=2}^{n} \sum_{j=1}^{m} w_j(s_i) f(x_j, s_{i-1})ds_1$$

let us consider

$$g(s_1) = \prod_{i=2}^{n} \sum_{j=1}^{m} w_j(s_i) f(x_j, s_{i-1})$$

again applying the above proposed methods of weights (w_j) and the corresponding nodes (x_j) in the above equation to the outer integral leads to

$$\int_a^b \prod_{i=2}^{n} \int_{c(s_i)}^{d(s_i)} f(s_i, s_1) ds_i ds_1 \approx \int_a^b g(s_1) ds_1 \approx \sum_{j=1}^{m} w_j g(x_j) \tag{29}$$

The domain of integration for the above methods is conventionally taken as [0, 1) except Gauss Legendre quadrature method and this method is taken as [-1, 1]. Sum of weights is a range of domain, n be the number of dimensions and m be the number of points. In this approach, the same number of points is considering to each integrals.

4. NUMERICAL EXAMPLES

The following examples are given to show the accuracy and efficiency of numerical integration by using Haar wavelet, Block-pulse function, Chebyshev wavelet and Since function are compared with Gauss Legendre quadrature method and exact solution. The absolute and relative error has been calculated and compared to each method for the given examples are shown in the below tables and to illustrate the computation performance of multiple integrals in Figure 1 to 6 obtained with above method.

Definition 4.1 *Suppose that* x_t *and* x_a *denote the true and approximate values of a integrals then the error incurred on approximating* x_t *by* x_a *is given by*

$$e = x_t - x_a$$

i.e. magnitude of the absolute error e_a is given by

$$e_a = |x_t - x_a| \tag{30}$$

Definition 4.2 *Relative Error or normalized error* e_r *in representing a true value* x_t *by an approximate value* x_a *is defined by*

$$e_r = \frac{|x_t - x_a|}{|x_t|} \tag{31}$$

4.1. Examples for Single Integration

Example 1

$$\int_0^{\sqrt[3]{\pi^2}} \sqrt{x} cos^2(x^{\frac{3}{2}}) dx$$

The solution of the integral is 4.531824108052927 Using eqn. (30). we calculated the absolute error, where x_a is the approximate value of the integral with the above method and compared with Gauss Legendre quadrature and exact solution are shown in Table 1. and using eqn. (31). we calculated the relative error and shown in Table 2. From the table as considering NN be the number of nodes. An illustrate the computation performance of integration in Figure 1. obtained with above method.

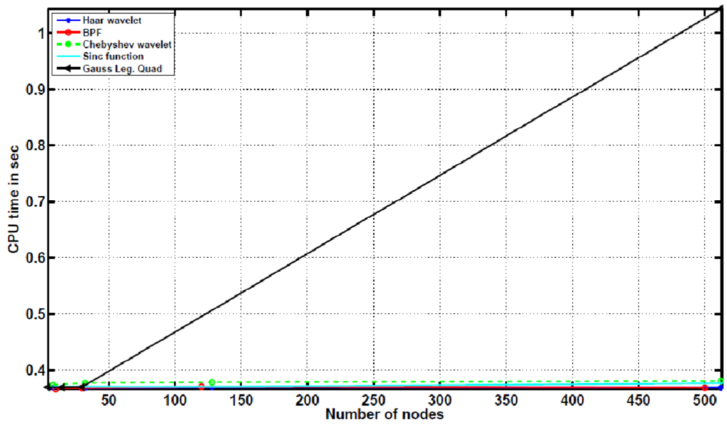

Figure 1: Time Evolution of Given Methods for Example 1

Table 1: The Absolute Error for Example 1

Haar Wavelet		BPF		Chebyshev Wavelet		Sinc Function		Gauss Leg. Quad.	
(J,NN)	e_a	(m,NN)	e_a	(k,M,NN)	e_a	(N,NN)	e_a	(n,NN)	e_a
2,8	4.0009E-01	10,10	1.8978E-01	3,2,8	4.0009E-01	4,9	8.4416E-01	4,4	6.4792E-03
4,32	1.4264E-02	30,30	1.6140E-02	4,4,32	1.0206E-02	30,61	3.4974E-03	15,15	3.4578E-04
6,128	1.1426E-03	120,120	1.2803E-03	6,4,128	2.6268E-04	64,129	9.4197E-05	30,30	4.7708E-05
8,512	1.0585E-04	500,500	1.1013E-04	8,4,512	3.5822E-05	256,513	8.8818E-16	512,512	9.9912E-09

Table 2: The Relative Error for Example 1

Haar Wavelet		BPF		Chebyshev Wavelet		Sinc Function		Gauss Leg. Quad.	
(J,NN)	e_r	(m,NN)	e_r	(k,M,NN)	e_r	(N,NN)	e_r	(n,NN)	e_r
2,8	8.8286E-02	10,10	4.1877E-02	3,2,8	8.8286E-02	4,9	1.8627E-01	4,4	1.4297E-03
4,32	3.1475E-03	30,30	3.5614E-03	4,4,32	2.2521E-03	30,61	7.7175E-04	15,15	7.6300E-05
6,128	2.5212E-04	120,120	12.8251E-04	6,4,128	5.7963E-05	64,129	2.0786E-05	30,30	1.0527E-05
8,512	2.3356E-05	500,500	2.4302E-05	8,4,512	7.9046E-06	256,513	1.9599E-16	512,512	2.2047E-09

Example 2

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{1}{(2007^x+1)} \frac{sin^{2008}x}{(sin^{2008}x+cos^{2008}x)}dx$$

The exact solution is $\frac{\pi}{4} \approx 0.7853981633974483.$ Using eqn. (30). we calculated the absolute error, where x_a is the approximate value of the integral with the above method and compared with Gauss Legendre quadrature and exact solution are shown in Table 3. and using eqn. (31). we calculated the relative error and shown in Table 4. From the table as considering NN be the number of nodes. An illustrate the computation performance of integration in Figure 2. obtained with above method.

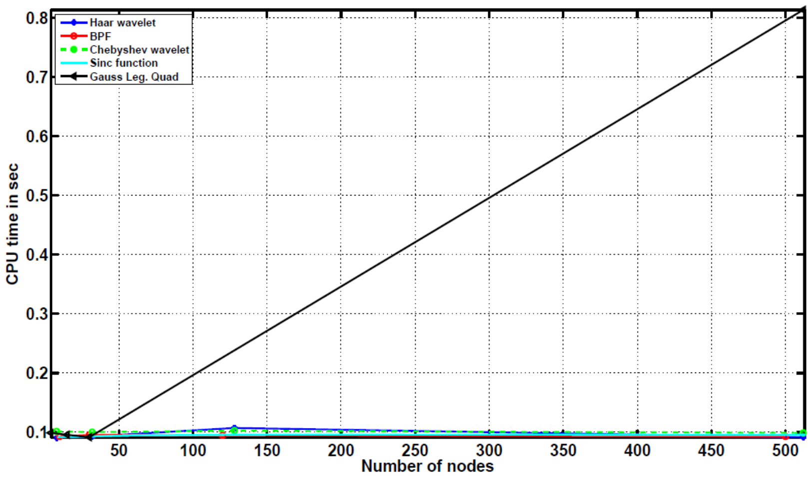

Figure 2: Time Evolution of Given Methods for Example 2

Table 3: The Absolute Error for Example 2

Haar Wavelet		BPF		Chebyshev Wavelet		Sinc funCtion		Gauss Leg. Quad.	
(J,NN)	e_a	(m,NN)	e_a	(k,M,NN)	e_a	(N,NN)	e_a	(n,NN)	e_a
2,8	0.0000E+00	10,10	2.4758E-14	3,2,8	0.0000E+00	4,9	1.6641E-01	4,4	2.3899E-01
4,32	0.0000E+00	30,30	8.2157E-15	4,4,32	1.1102E-16	30,61	1.4478E-01	15,15	2.2177E-02
6,128	0.0000E+00	120,120	2.2204E-16	6,4,128	0.0000E+00	64,129	6.8996E-02	30,30	1.1467E-02
8,512	3.3307E-16	500,500	2.2204E-16	8,4,512	2.2204E-16	256,513	1.0589E-02	512,512	2.9933E-03

Table 4: The Relative Error for Example 2.!

Haar Wavelet		BPF		Chebyshev Wavelet		Sinc Function		Gauss Leg. Quad.	
(J,NN)	e_r	(m,NN)	e_r	(k,M,NN)	e_r	(N,NN)	e_r	(n,NN)	e_r
2,8	0.0000E+00	10,10	3.1523E-14	3,2,8	0.0000E+00	4,9	12.1188E-01	4,4	3.0429E-01
4,32	0.0000E+00	30,30	1.0460E-14	4,4,32	1.4136E-16	30,61	1.8434E-01	15,15	2.8237E-02
6,128	0.0000E+00	120,120	2.8272E-16	6,4,128	0.0000E+00	64,129	8.7848E-02	30,30	1.4601E-02
8,512	4.2407E-16	500,500	2.8272E-16	8,4,512	2.8272E-16	256,513	1.3482E-02	512,512	3.8112E-03

4.2. Examples for Double Integration

Example 3

$$\int_0^1 \int_0^1 \frac{1}{1-xy} dxdy$$

The exact solution is $\sum_{n=1}^{\infty} \frac{1}{n^2} \approx 1.644934066848226$. Using eqn. (30). we calculated the absolute error, where x_a is the approximate value of the integral with the above method and compared with Gauss Legendre quadrature and exact solution are shown in Table 5. and using eqn. (31). we calculated the relative error and shown in Table 6. From the table as considering NN be the number of nodes. An illustrate the computation performance of integration in Figure 3. obtained with above method.

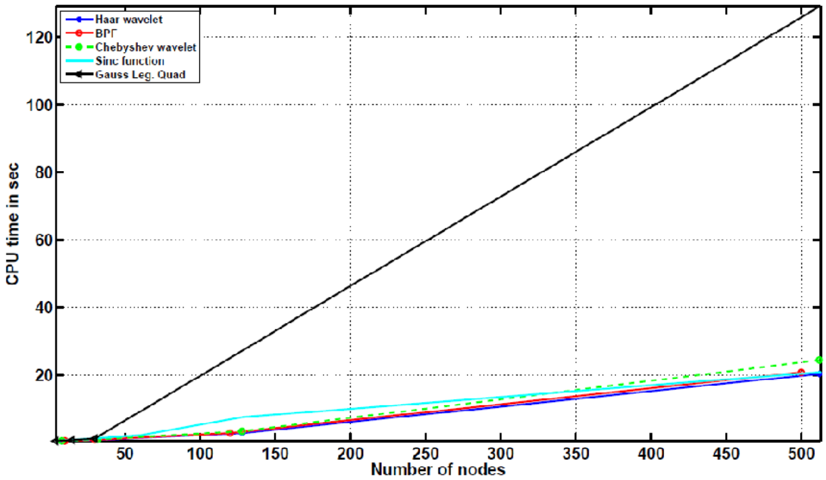

Figure 3: Time Evolution of Given Methods for Example 3

Table 5: The Absolute Error for Example 3

Haar Wavelet		BPF		Chebyshev Wavelet		Sinc Function		Gauss Leg. Quad.	
(J,NN)	e_a	(m,NN)	e_a	(k,M,NN)	e_a	(N,NN)	e_a	(n,NN)	e_a
2,8	5.7222E-02	10,10	4.6446E-02	3,2,8	5.7222E-02	4,9	1.2976E-02	4,4	3.8940E-02
4,32	1.5186E-02	30,30	1.6173E-02	4,4,32	1.1349E-02	30,61	9.3798E-07	15,15	3.3315E-03
6,128	3.8718E-03	120,120	4.1279E-03	6,4,128	2.8602E-03	64,129	5.2915E-10	30,30	8.6129E-04
8,512	9.7397E-04	500,500	9.9729E-04	8,4,512	7.1653E-04	125,251	3.6193E-14	512,512	3.1996E-06

Table 6: The Relative Error for Example 3

Haar Wavelet		BPF		Chebyshev Wavelet		Sinc Function		Gauss Leg. Quad.	
(J,NN)	e_r	(m,NN)	e_r	(k,M,NN)	e_r	(N,NN)	e_r	(n,NN)	e_r
2,8	3.4787E-02	10,10	2.8236E-02	3,2,8	3.4787E-02	4,9	7.8883E-03	4,4	2.3673E-02
4,32	9.2320E-03	30,30	9.8321E-03	4,4,32	6.8992E-03	30,61	5.7022E-07	15,15	2.0253E-03
6,128	2.3538E-03	120,120	2.5094E-03	6,4,128	1.7388E-03	64,129	3.2169E-10	30,30	5.2360E-04
8,512	5.9210E-04	500,500	6.0628E-04	8,4,512	4.3560E-04	125,251	2.2003E-14	512,512	1.9451E-06

Example 4

$$\int_0^2 \int_{x^2}^4 \frac{x^3}{\sqrt{x^4+y^2}}\,dxdy$$

The exact solution is $4(\sqrt{2}-1) \approx 1.656854249492381$. Using eqn. (30). we calculated the absolute error, where x_a is the approximate value of the integral with the above method and compared with Gauss Legendre quadrature and exact solution are shown in Table 7. and using eqn. (31). we calculated the relative error and shown in Table 8. From the table as considering NN be the number of nodes. An illustrate the computation performance of integration in Figure 4. obtained with above method.

Table 7: The Absolute Error for Example 4

Haar Wavelet		BPF		Chebyshev Wavelet		Sinc Function		Gauss Leg. Quad.	
(J,NN)	e_a	(m,NN)	e_a	(k,M,NN)	e_a	(N,NN)	e_a	(n,NN)	e_a
2,8	9.6835E-03	10,10	5.9816E-03	3,2,8	9.6835E-03	3,7	8.4328E-03	4,4	7.8592E-04
3,16	2.1421E-03	16,16	2.1421E-03	3,3,12	6.4107E-04	4,9	9.7525E-04	8,8	1.4282E-04
4,32	4.5917E-04	32,32	4.5917E-04	3,4,16	4.7310E-04	6,13	4.3953E-04	12,12	2.9766E-05
5,64	9.5168E-05	64,64	9.5168E-05	4,4,32	1.4638E-04	7,15	3.3811E-04	16,16	9.4974E-06

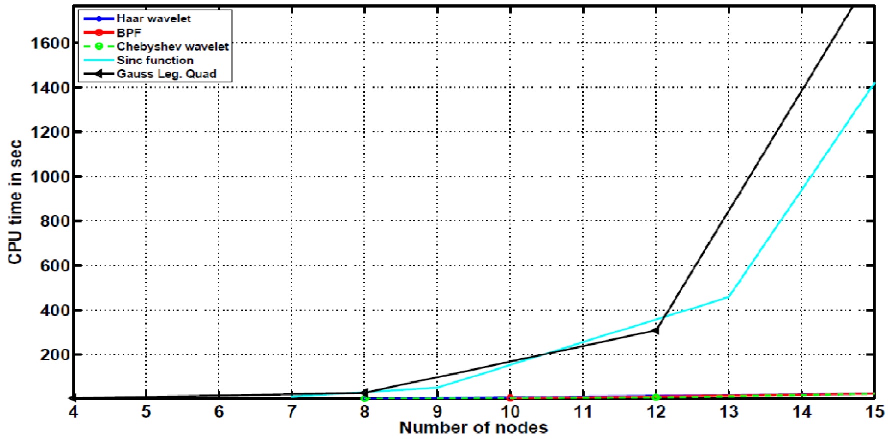

Figure 4: Time Evolution of given Methods for Example 4

Table 8: The Relative Error for Example 4

Haar Wavelet		BPF		Chebyshev Wavelet		Sinc Function		Gauss Leg. Quad.	
(J,NN)	e_r	(m,NN)	e_r	(k,M,NN)	e_r	(N,NN)	e_r	(n,NN)	e_r
2,8	5.8445E-03	10,10	3.6102E-03	3,2,8	5.8445E-03	3,7	5.0896E-03	4,4	4.7435E-04
3,16	1.2929E-03	16,16	1.2929E-03	3,3,12	3.8692E-04	4,9	5.8861E-04	8,8	8.6197E-05
4,32	2.7713E-04	32,32	2.7713E-04	3,4,16	2.8554E-04	6,13	2.6528E-04	12,12	1.7965E-05
5,64	5.7439E-05	64,64	5.7439E-05	4,4,32	8.8347E-05	7,15	2.0407E-04	16,16	5.7322E-06

4.3. Examples for Triple Integration

Example 5

$$\int_0^1 \int_0^1 \int_0^1 \frac{1}{1-xyz}\,dxdydz$$

The exact solution is $\sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.202056903159594$. Using eqn. (30). we calculated the absolute error, where x_a is the approximate value of the integral with the above method and compared with Gauss Legendre quadrature and exact solution are shown in Table 9. and using eqn. (31). we calculated the relative error and shown in Table 10. From the table as considering NN be the number of nodes. An illustrate the computation performance of integration in Figure 5. obtained with above method.

Table 9: The Absolute Error for Example 5

Haar Wavelet		BPF		Chebyshev Wavelet		Sinc Function		Gauss Leg. Quad.	
(J,NN)	e_a	(m,NN)	e_a	(k,M,NN)	e_a	(N,NN)	e_a	(n,NN)	e_a
2,8	4.0677E-03	10,10	2.8589E-03	3,2,8	4.0677E-03	3,7	1.0096E-02	8,8	8.8419E-05
3,16	1.3319E-03	16,16	1.3319E-03	3,4,16	4.5409E-04	7,15	1.0068E-03	16,16	6.3044E-06
4,32	4.1445E-04	32,32	4.1445E-04	4,4,32	1.1999E-04	16,33	1.8211E-05	32,32	4.2024E-07
5,64	1.2435E-04	64,64	1.2435E-04	4,8,64	4.0084E-06	32,65	1.1358E-07	64,64	2.7112E-08

Table 10: The Relative Error for Example 5

Haar Wavelet		BPF		Chebyshev Wavelet		Sinc Function		Gauss Leg. Quad.	
(J,NN)	e_r	(m,NN)	e_r	(k,M,NN)	e_r	(N,NN)	e_r	(n,NN)	e_r
2,8	3.3839E-03	10,10	2.3783E-03	3,2,8	3.3839E-03	3,7	8.3988E-03	8,8	7.3556E-05
3,16	1.1081E-03	16,16	1.1081E-03	3,4,16	3.7776E-04	7,15	8.3758E-04	16,16	5.2447E-06
4,32	3.4479E-04	32,32	3.4479E-04	4,4,32	9.9824E-05	16,33	1.5150E-05	32,32	3.4960E-07
5,64	1.0345E-04	64,64	1.0345E-04	4,8,64	3.3346E-06	32,65	9.4487E-08	64,64	2.2555E-08

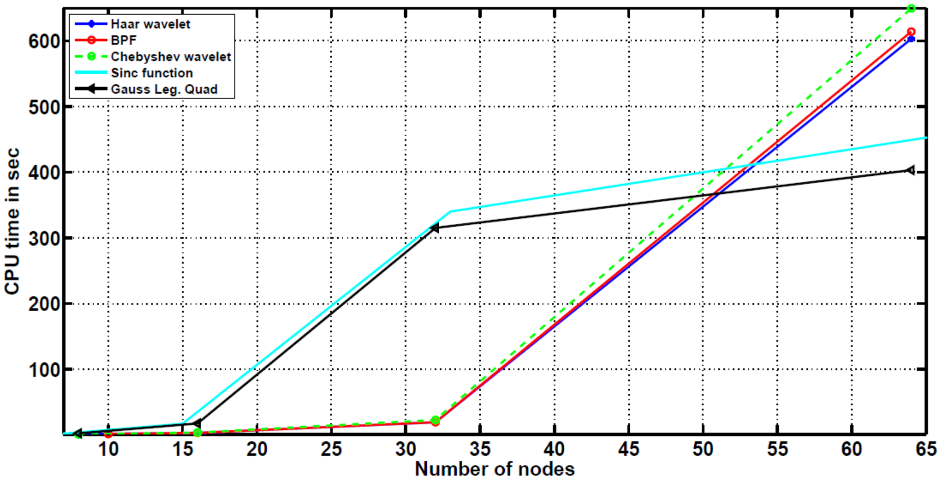

Figure 5: Time Evolution of Given Methods for Example 5

Example 6

$$\int_0^2 \int_0^{\sqrt{4-z^2}} \int_0^{\sqrt{4-y^2-z^2}} (x^2+y^2+z^2)dxdydz$$

The exact solution is $\frac{16\pi}{3} \approx 10.053096491487338$. Using eqn. (30). we calculated the absolute error, where x_a is the approximate value of the integral with the above method and compared with Gauss Legendre quadrature and exact solution are shown in Table 11. and using eqn. (31). we calculated the relative error and shown in Table 12. From the table as considering NN be the number of nodes. An illustrate the computation performance of integration in Figure 6. obtained with above method.

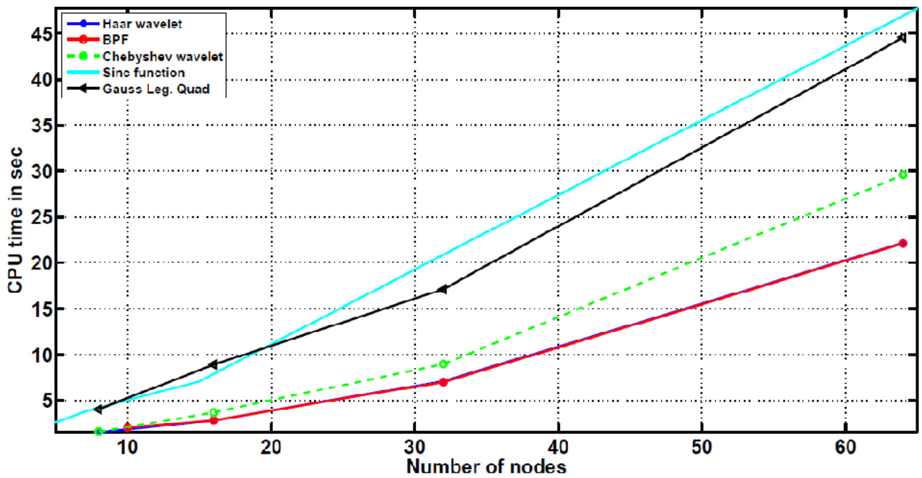

Figure 6: Time Evolution of Given Methods for Example 6

Table 11: The Absolute Error for Example 6

Haar Wavelet		BPF		Chebyshev Wavelet		Sinc Function		Gauss Leg. Quad.	
(J,NN)	e_a	(m,NN)	e_a	(k,M,NN)	e_a	(N,NN)	e_a	(n,NN)	e_a
2,8	9.7155E-02	10,10	6.8570E-02	3,2,8	9.7155E-02	2,5	5.7273E-02	8,8	5.1733E-03
3,16	3.2971E-02	16,16	3.2971E-02	3,4,16	1.5726E-02	3,7	4.1350E-02	16,16	6.9555E-04
4,32	1.1262E-02	32,32	1.1262E-02	4,4,32	5.4036E-03	7,15	6.0570E-03	32,32	9.0621E-05
5,64	3.8747E-03	64,64	3.8747E-03	4,8,64	7.7776E-04	32,65	6.5828E-07	64,64	1.1580E-05

Table 12: The Relative Error for Example 6

Haar Wavelet		BPF		Chebyshev Wavelet		Sinc Function		Gauss Leg. Quad.	
(J,NN)	e_r	(m,NN)	e_r	(k,M,NN)	e_r	(N,NN)	e_r	(n,NN)	e_r
2,8	9.6642E-03	10,10	6.8208E-03	3,2,8	9.6642E-03	2,5	5.6971E-03	8,8	5.1459E-04
3,16	3.2797E-03	16,16	3.2797E-03	3,4,16	1.5643E-03	3,7	4.1131E-03	16,16	6.9188E-05
4,32	1.1203E-03	32,32	1.1203E-03	4,4,32	5.3750E-04	7,15	6.0250E-04	32,32	9.0143E-06
5,64	3.8543E-04	64,64	3.8543E-04	4,8,64	7.7366E-05	32,65	6.5481E-08	64,64	1.1519E-06

5. RESULTS AND DISCUSSIONS

Maximum absolute and relative errors obtained by the numerical multiple integrations based on the given methods for certain problems are shown in Tables 1 to 12. Comparison of maximum absolute and relative errors with Gaussian Legendre quadrature method and exact solution determines the accuracy of given methods. Estimates the computation performance for the certain problems is shown in figure 1 to 6. Comparison of time evolutions with Gaussian Legendre quadrature method. It is clearly seen that numerical multiple integrations using above methods have produced improved results ,as well as computation performance, are compared with Gaussian Legendre quadrature method for numerical multiple integrations.

6. CONCLUSIONS

In this paper, Haar wavelet, Block-pulse function, Chebyshev wavelet and Sinc functions were applied for numerical multiple integrations (double, triple integrals) with variable and constant limits. Compared between these methods with Gaussian Legendre quadrature method. The exact solution clearly shows that the above methods and given a much better (anyone of Haar wavelet, Block-pulse function, Chebyshev wavelet and Sinc functions and it depends on the problem) results as well as computation performance than Gaussian Legendre quadrature method.

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